# Truthmakers[1]

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[1] Slides marked with $*$ were not covered during the lecture, but they are included for completeness. You can skip them for what concerns the exam material of the course.

# Readings

**Suggested:**

- Chapter 5 in lecture notes.
- Rodriguez-Pereyra, G. (2006). Truthmakers. *Philosophy Compass*, 1(2), 186-200.
- van Fraassen, B. C. (1969). Facts and tautological entailments. *The Journal of Philosophy*, 66(15), 477-487.
- Fine, K. (2017). A theory of truthmaker content I: Conjunction, disjunction and negation. *Journal of Philosophical Logic*, 46(6), 625-674.
- Fine, K. (2017). Truthmaker semantics. *A Companion to the Philosophy of Language*, 556-577.
- Fine, K., & Jago, M. (2019). Logic for exact entailment. *The review of symbolic logic*, 12(3), 536-556.

# Plan

1. Truthmakers
2. Truthmaker Semantics
3. Fine$^*$
4. Exact Truthmaking

# Outline

# What is a truthmaker?

- Consider some ordinary sentences:
  1. *The grass is green.*
  2. *I am here.*
  3. *Dinosaurs existed.*
  4. *Unicorns do not exist.*
- All of these are (or at least seem to be) true.
- But what in reality **makes** them true?
- Is 'I am here' true because I am here in the world, or does my being here somehow depend on the truth of the sentence?
- Truthmaker theory takes seriously the idea that **the world makes our true sentences true**: the direction of explanation goes from world to word.

# Truthbearers and truthmakers

- **Truthbearer**: something that can be true or false (sentences, beliefs, judgements, propositions, ...).
- **Truthmaker**: something in the world *in virtue of which* a truthbearer is true.
- Example:
  - Truthbearer: 'The cat is on the mat.'
  - Truthmaker: that very cat's being on that very mat.
- Truthmaker theorists hold (roughly): whenever a truthbearer is true, there is some portion of reality that makes it true.
- What kinds of things can be truthmakers, and do *all* truths have them?

# Why care about truthmakers?

- Truthmaker theory sharpens the old correspondence idea: true thoughts *match* reality.
- It forces us to connect:
  - claims about truth (logic, semantics),
  - with claims about what there is (metaphysics, ontology).
- Often used as an *argument for* or *against* certain entities:
  - If moral truths are real, what in the world makes them true?
  - Do modal truths ('It could have been otherwise') require possible worlds as truthmakers?

# Truthmaking as entailment

- A tempting idea: connect truthmaking with logical entailment.

> If $x$ makes $p$ true and $p \models q$, then $x$ also makes $q$ true.

- Consider $p =$ 'Amsterdam is in the Netherlands', and let $x$ be *the fact that you are at the lecture.*
- $p \vee \neg p$ is always true.
- On a naive view, *any* existing $x$ could serve as a truthmaker for $p \vee \neg p$.
- If we also assume that a disjunction is made true only by one of its disjuncts, we are pushed to say that your merely being here makes it true that $p$.
- That seems wrong: your presence is irrelevant to whether Amsterdam is in the Netherlands.
- This motivates more refined notions such as *relevant* entailment (more on this in Robert's lecture). [Restall 1996]

# Truthmaking as necessitation

- From a more metaphysical point of view, we can characterise truthmaking via **necessitation**:

> An entity $T$ is a truthmaker for a proposition $p$ iff
> $T$ *necessitates* $p$ being true.

- So once the truthmaker is in place, the truth could not have failed.
- This builds into truthmaking an explanatory order: truthmakers are prior to (and more fundamental than) the truths they make.
- It also gives us a handle on what kinds of entities can plausibly serve as truthmakers.

# Truthmakers as objects

- A first, naive view is to identify truthmakers with ordinary objects.
- Example: ‘Socrates exists’ is made true by Socrates himself.
- But under a strict necessitation condition, a bare particular rarely suffices:
  - The rose could exist without being red.
  - So the rose alone does not necessitate the truth of ‘The rose is red’.
- Conclusion: if we want necessitating truthmakers, we usually need entities *more fine-grained* than just objects.

# Truthmakers as states of affairs

- Many truthmaker theorists prefer states of affairs as truthmakers.
- Rough idea: a state of affairs has the form

$$\langle \text{object(s)}, \text{property/relation} \rangle$$

  actually instantiating that property or relation.
- The truth of 'The rose is red' is made true by the state of affairs:

  the rose's being red.
- The truth of 'John greeted Mary' is made true by the relational state of affairs:

  John's greeting Mary.
- Unlike bare objects, such states of affairs cannot exist without the corresponding sentence being true, so they are good candidates for necessitating truthmakers.

# Other candidates

- Besides states of affairs, philosophers have proposed:
  - **Events**: particularly for truths about what happens or changes.
  - **Tropes**: particularised properties (e.g. this very redness) as truthmakers for predicative truths.
  - ...
- For today we will mostly talk in the neutral vocabulary of states of affairs/facts.

# Which truths have truthmakers?

- Two broad positions:
- **Maximalism**: every truth has a truthmaker (Armstrong 2004).
- **Non-maximalism**: some truths lack truthmakers; there are *truthmaker gaps* (e.g. Cameron 2008).
- Maximalists get a very tight connection between truth and reality, but they also inherit demanding cases:
  - negative truths,
  - general and universal truths,
  - modal, mathematical, moral truths,
  - liar-style sentences.
- Non-maximalists avoid some metaphysical costs by denying that all truths need truthmakers.

# Challenges to maximalism: hard truths

- Maximalism has intuitive appeal: truth is always grounded in how things are.
- But some kinds of truths look especially problematic:
- **Negative truths**
  - ‘Amsterdam is not in Italy.’
- **Non-existence truths**
  - ‘There are no unicorns.’
  - ‘Pegasus does not exist.’
- What in the world could *make* such truths true?

# Strategies for negative truths

- Two influential maximalist strategies:
- **Negative facts**
  - For each true negative sentence, there is a corresponding negative fact.
  - Example: the fact that there are no unicorns.
  - Objection: this populates our ontology with many strange absence-like entities.
- **Totality facts**
  - Start with all the positive facts about what exists and how it is.
  - Add a *totality* fact saying: and that is everything.
  - From this, it follows that there are no further things (such as unicorns).
  - Objection: totality facts are metaphysically heavyweight.

# A liar-style challenge to maximalism

Let $M$ be the sentence:

'This sentence has no truthmaker.'

- Milne (2005) argues:
- Suppose, for reductio, that $M$ has a truthmaker $T$.
- Then $M$ is true. So what $M$ says is the case is the case: $M$ has no truthmaker.
- On the supposition that $M$ has a truthmaker, we deduce that $M$ has *no* truthmaker.
- By reductio ad absurdum, $M$ has no truthmaker.
- But that is exactly what $M$ says. So $M$ is true and yet has no truthmaker.
- Hence $M$ is (apparently) a truth without a truthmaker: a counterexample to maximalism, analogous to liar-style paradoxes for truth.

# Outline

# From metaphysics to semantics

Bas van Fraassen

Kit Fine

- So far, we have treated truthmaker theory as a **metaphysical** project:
  - what in reality makes a given sentence, belief, or proposition true?
  - which entities we must admit to be truthmakers?
- But there is also a more **formal** way of thinking about truthmakers:
  - build truthmakers *directly into the semantics* of a language,
  - so that the semantic value of a sentence tells us not just *when* it is true, but *what makes it true*.
- This brings us to what is now known as **truthmaker semantics**, first introduced by Bas van Fraassen and recently further developed and studied by Kit Fine.

# Hyperintensionality

- In an **extensional** semantics, the meaning of a sentence is just its truth value in a model (relative to an assignment).
- **Intensional** semantics improves on this: meanings are functions from possible worlds to truth values (sets of worlds).
- But this is still not *fine-grained* enough.
- Consider:

  (1) Venus is Venus.

  (2) Venus is the morning star.

  (1) and (2) are necessarily equivalent (true in exactly the same worlds).
- Intensional semantics cannot distinguish them, but they clearly differ in *informational content* and in what would count as a truthmaker.
- We need a **hyperintensional** semantics: one that distinguishes necessarily equivalent sentences.

[Berto and Nolan 2023]

# A note on formalisms and notation

- We will see a variety of ways to formalize truthmaker semantics.
- The point is not to confuse you. Rather, you should get used to the fact that the literature (not just on truthmaker semantics) exhibits:
  - different notations,
  - slightly different but equivalent formalizations,
  - different names for the same underlying notions.
- A useful skill:
  - be able to navigate between these presentations,
  - recognize when they amount to the *same* formalism,
  - and see which parts are notational choice vs. substantive.
- We start with the truthmaker semantics proposed by Bas van Fraassen (1969)[2]

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[2] The presentation that follows is an adaptation of Bas van Fraassen (1969) as well as some 2020 posts in Bas van Fraassen blog.

# Facts and their combination

We work with a countable set of *facts* Fct.

- Some facts are *atomic*.
- Other facts are *complex*, formed by *combination*.

There is a binary operation '$\cdot$' on facts:

$$b, c \in \mathsf{Fct} \ \Rightarrow \ b \cdot c \in \mathsf{Fct}.$$

It satisfies:

$$\begin{aligned} b \cdot b &= b \quad \text{(idempotent)} \\ b \cdot c &= c \cdot b \quad \text{(commutative)} \\ b \cdot (c \cdot d) &= (b \cdot c) \cdot d \quad \text{(associative)} \end{aligned}$$

# Atomic and complex facts

**Facts:** Every fact is a (possibly infinite) combination of atomic facts, and two facts are identical iff they are combinations of exactly the same atomic facts.

**Atomic fact:** A fact $b$ is *atomic* iff for every fact $c$,

$$b = b \cdot c \;\Rightarrow\; b = c$$

- Complex facts are like mereological sums or “bundles” of atomic facts.
- The order of components in a bundle is irrelevant.

# Subordination of facts

For facts $b, c \in \mathsf{Fct}$:

$$b \leq c \quad \text{iff} \quad \exists d \in \mathsf{Fct} \text{ such that } b = d \cdot c$$

$b \leq c$ means: $b$ is at least as *informative* as $c$, because $b$ contains all the atomic facts that $c$ contains (and maybe more).[3]

- $\leq$ is reflexive:
$$b = b \cdot b \;\Rightarrow\; b \leq b$$
- $\leq$ is transitive: if $b \leq c$ and $c \leq e$ then $b \leq e$.
- $\leq$ is antisymmetric (recall identity of facts): if $a \leq b$ and $b \leq a$, then $a = b$.
- So it is a partial order.

[3] Keep in mind that we will reverse the conceptual order of $b \leq c$ when moving to Fine's formalization.

# Propositions as closed sets of facts

For $X \subseteq \mathsf{Fct}$, we define its closure under $\leq$ as:

$$[X] \; = \; \{y \in \mathsf{Fct} : \exists x \in X \text{ with } y \leq x\}$$

A set $Q \subseteq \mathsf{Fct}$ is a *proposition* iff it is closed under $\leq$, i.e.

$$Q = [Q]$$

The relation $e \triangleright Q$ (*e makes Q true*) is defined by:

$$e \triangleright Q \quad \text{iff} \quad e \in Q$$

If $e \triangleright Q$ and $f \leq e$, then $f \triangleright Q$ as well, by closure.

# The lattice of propositions*

Let Prop be the set of all propositions.

- Partial order: $Q \leq R$ iff $Q \subseteq R$
- Meet: $Q \wedge R = Q \cap R$
- Join: $Q \vee R = [Q \cup R]$

Any intersection of propositions is a proposition, and any union has a closure. Thus $(\mathsf{Prop}, \leq)$ is a complete lattice.

# Special structure of this lattice*

In this specific construction, something stronger holds.

For any propositions $Q, R$:

$$Q \cap R = \{e \cdot f : e \in Q,\ f \in R\}$$

For any propositions $Q, R$:

$$[Q \cup R] = Q \cup R$$

So *every union of two propositions is already closed*.

Thus, for propositions:

$$Q \wedge R = Q \cap R \qquad Q \vee R = Q \cup R$$

Hence the lattice Prop is distributive.

# Complementation on atomic facts

On the set of atomic facts there is an operation $b \mapsto \bar{b}$ such that:

- $\bar{b}$ is atomic,
- $\bar{\bar{b}} = b$ for every atomic fact $b$.

Think of $b$ as "the apple is red", and $\bar{b}$ as "the apple is non-red".

Complex facts *do not* in general have complements (there are no disjunctive facts in the ontology).

Hence, there is no global Boolean complement on the set of propositions $\mathsf{Prop}$. Negation must be handled indirectly, via the propositions expressed by sentences.

# Language and primary bases

Fix a propositional language with connectives $\neg, \wedge, \vee$.

For each atomic sentence $p$, pick an atomic fact $e$:

$$\mathsf{T}(p) = \{e\} \qquad \mathsf{F}(p) = \{\overline{e}\}$$

For $X, Y \subseteq \mathsf{Fct}$ we define:

$$X \cdot Y \;=\; \{e \cdot f : e \in X,\; f \in Y\}$$

# Recursive definition of T- and F-bases

For all sentences $A, B$:

$$\mathsf{T}(\neg A) = \mathsf{F}(A) \qquad \mathsf{F}(\neg A) = \mathsf{T}(A)$$

$$\mathsf{T}(A \wedge B) = \mathsf{T}(A) \cdot \mathsf{T}(B) \qquad \mathsf{F}(A \wedge B) = \mathsf{F}(A) \cup \mathsf{F}(B)$$

$$\mathsf{T}(A \vee B) = \mathsf{T}(A) \cup \mathsf{T}(B) \qquad \mathsf{F}(A \vee B) = \mathsf{F}(A) \cdot \mathsf{F}(B)$$

- $\mathsf{T}(A)$: the set of primary facts that make $A$ true.
- $\mathsf{F}(A)$: the set of primary facts that make $A$ false.

# Examples

Fix atomic sentences $p, q, r$ with basic facts $e_p, e_q, e_r$ and complements $\overline{e_p}, \overline{e_q}, \overline{e_r}$. Then

$$\mathsf{T}(p) = \{e_p\},\ \mathsf{F}(p) = \{\overline{e_p}\}, \quad \ldots$$

(For sets $X, Y$ of facts, $X \cdot Y = \{e \cdot f : e \in X, f \in Y\}$.)

1. $p \wedge \neg q$ and $\neg p \vee q$

$$\mathsf{T}(p \wedge \neg q) = \{e_p \cdot \overline{e_q}\} \quad \mathsf{F}(p \wedge \neg q) = \{\overline{e_p},\ e_q\}$$

$$\mathsf{T}(\neg p \vee q) = \{\overline{e_p},\ e_q\} \quad \mathsf{F}(\neg p \vee q) = \{e_p \cdot \overline{e_q}\}$$

**2.** $A = (p \wedge q) \vee \neg r$

$$\mathsf{T}(A) = \{e_p \cdot e_q,\ \overline{e_r}\} \qquad \mathsf{F}(A) = \{\overline{e_p} \cdot e_r,\ \overline{e_q} \cdot e_r\}$$

**3.** $B = (p \wedge q) \vee (p \wedge r)$**,** $C = p \wedge (q \vee r)$

$$\mathsf{T}(B) = \{e_p \cdot e_q,\ e_p \cdot e_r\} \ = \ \mathsf{T}(C)$$

# From bases to propositions of sentences

For each sentence $A$, we determine two propositions:

$$\mathsf{T}^*(A) \ = \ [\mathsf{T}(A)] \qquad \mathsf{F}^*(A) \ = \ [\mathsf{F}(A)]$$

So $\mathsf{T}^*(A)$ is the proposition (closed set of facts) that $A$ is true, and $\mathsf{F}^*(A)$ the proposition that $A$ is false.

(Later we will identify $\mathsf{T}(A)$ with the set of *exact* truthmakers of $A$, and $\mathsf{T}^*(A)$ with the set of *inexact* truthmakers of $A$)

We have:

$$\mathsf{T}^*(A \wedge B) = \mathsf{T}^*(A) \cap \mathsf{T}^*(B) \qquad \mathsf{T}^*(A \vee B) = \mathsf{T}^*(A) \cup \mathsf{T}^*(B)$$

$$\mathsf{F}^*(A \wedge B) = \mathsf{F}^*(A) \cup \mathsf{F}^*(B) \qquad \mathsf{F}^*(A \vee B) = \mathsf{F}^*(A) \cap \mathsf{F}^*(B)$$

# Examples

Fix atomic sentences $p, q, r$ with basic facts $e_p, e_q, e_r$ and complements $\overline{e_p}, \overline{e_q}, \overline{e_r}$. For facts $f, g$, recall $f \leq g$ for "$f$ is subordinate to $g$". Then:

1. Atoms and their negations

$$\mathsf{T}^*(p) = [\{e_p\}] = \{f : f \leq e_p\} \qquad \mathsf{T}^*(\neg p) = [\{\overline{e_p}\}] = \{f : f \leq \overline{e_p}\}$$

2. Conjunction and disjunction

$$\mathsf{T}(p \wedge q) = \{e_p \cdot e_q\} \;\Rightarrow\; \mathsf{T}^*(p \wedge q) = [\{e_p \cdot e_q\}] = \{f : f \leq e_p \cdot e_q\}$$

$$\mathsf{T}(p \vee q) = \{e_p, e_q\} \;\Rightarrow\; \mathsf{T}^*(p \vee q) = [\{e_p, e_q\}] = \{f : f \leq e_p \text{ or } f \leq e_q\}$$

3. A more complex formula

$$A = (p \wedge q) \vee \neg r$$

From $\mathsf{T}(A) = \{e_p \cdot e_q,\ \overline{e_r}\}$ we get

$$\mathsf{T}^*(A) = [\{e_p \cdot e_q,\ \overline{e_r}\}] = \{f : f \leq e_p \cdot e_q \text{ or } f \leq \overline{e_r}\}$$

# Facts about double negation and order*

For every sentence $A$:

$$\mathsf{T}^*(\neg\neg A) = \mathsf{T}^*(A) \qquad \mathsf{F}^*(\neg\neg A) = \mathsf{F}^*(A).$$

Indeed, $\mathsf{T}(\neg\neg A) = \mathsf{F}(\neg A) = \mathsf{T}(A)$ and $\mathsf{F}(\neg\neg A) = \mathsf{T}(\neg A) = \mathsf{F}(A)$, so their closures coincide:

$$\mathsf{T}^*(\neg\neg A) = [\mathsf{T}(\neg\neg A)] = [\mathsf{T}(A)] \quad \mathsf{F}^*(\neg\neg A) = [\mathsf{F}(\neg\neg A)] = [\mathsf{F}(A)]$$

If $\mathsf{T}^*(A) \subseteq \mathsf{T}^*(B)$, then

$$\mathsf{T}^*(\neg B) \subseteq \mathsf{T}^*(\neg A)$$

If every truth-maker for $A$ is also a truth-maker for $B$, then every truth-maker for $\neg B$ is also a truth-maker for $\neg A$.

# Outline

# State Space*

van Fraassen formulation identifies a proposition with a closed set of facts. Fine (2017) instead takes a proposition to be any set of facts and further discusses different notions of propositions.

There is a non-empty set $S$ of *states* (facts, events, conditions, ...), partially ordered by parthood $\leq$ and closed under *fusion*:

$$s_1, s_2, \dots \in S \;\Rightarrow\; s_1 \sqcup s_2 \sqcup \dots \in S$$

We work with a structured state space

$$\langle S, \leq, \sqcup \rangle$$

Distinguished extremal states:

- *null* state $0$: fusion of no states; $0 \leq s$ for all $s \in S$,
- *full* state $1$: fusion of all states; $s \leq 1$ for all $s \in S$.

# Propositions and constraints*

A (unilateral) **proposition** is a set of verifying states:

$$P \subseteq S$$

Two propositions are identical iff they have the same verifiers.

Fine imposes structural conditions on such $P$:

- *Verifiability*: every proposition has at least one verifier:

$$P \neq \varnothing$$

- *Closure*: if $s_1, s_2, \dots \in P$, then their fusion is in $P$:

$$s_1 \sqcup s_2 \sqcup \dots \in P$$

- *Convexity*: if $s, u \in P$ and $s \leq t \leq u$, then $t \in P$

Different choices of constraints give different **domains** of propositions:

- **Full** domain: no structural constraints.
- **Semi-regular** domain: convexity only.
- **Regular** domain: closure + convexity (often also verifiability).

# Regular propositions and subject-matter*

In a **regular** domain, each non-empty proposition $P \subseteq S$ is:

- *closed*: closed under (finite / arbitrary) fusions.
- *convex*: if $s, u \in P$ and $s \leq t \leq u$, then $t \in P$.

Such a $P$ has a **maximal verifier**

$$p = \bigsqcup P$$

the fusion of all its verifiers. This $p$ is the *subject-matter* of $P$: the largest state within which $P$ is evaluated.

Let the *base verifiers* of $P$ be the $\leq$-minimal elements of $P$:

$$L(P) = \{p_i \in P : \neg\exists t \in P \; (t < p_i)\}$$

In a regular domain we can recover $P$ from $L(P)$ and $p$:

$$P = \{t \in S : \exists p_i \in L(P) \; (p_i \leq t \leq p)\}$$

This gives a restricted **monotonicity inside the subject-matter**:

$$s \in P \;\&\; s \leq t \leq p \;\Rightarrow\; t \in P$$

Verifiers must be “stable”: not below any $p_i$, and not above $p$.

# Boolean operations on unilateral propositions*

We work first with *unilateral* propositions, identified with sets of verifiers $P, Q \subseteq S$.

**Conjunction**: exact fusion of verifiers:

$$P \wedge Q \ = \ \{p \sqcup q : p \in P,\ q \in Q\}$$

A state verifies $P \wedge Q$ iff it is (exactly) a fusion of a verifier of $P$ and a verifier of $Q$.

**Disjunction**: pool verifiers and then close as required by the chosen domain:

| | |
|---|---|
| $P \vee Q = P \cup Q$ | (full domain) |
| $P \vee Q = (P \cup Q)^*$ | (semi-regular: convex closure) |
| $P \vee Q = (P \cup Q)^{**}$ | (regular: convex $+$ fusion closure) |

So $\wedge$ corresponds to *mereological fusion* of truthmakers. $\vee$ corresponds to *choice* between them, plus structural closure.

# Bilateral propositions and negation*

- A **bilateral proposition** is an ordered pair

$$P = (P^+, P^-)$$

  where $P^+ \subseteq S$ are its verifiers and $P^- \subseteq S$ its falsifiers.
- We extend the structural constraints componentwise ($P$ is regular, semi-regular, etc. iff both $P^+$ and $P^-$ are.)
- **Negation** just swaps positive and negative content:

$$\neg(P^+, P^-) = (P^-, P^+)$$

- For conjunction and disjunction we combine the unilateral operations on both sides:

$$(P^+, P^-) \wedge (Q^+, Q^-) = (P^+ \wedge Q^+,\ P^- \vee Q^-)$$
$$(P^+, P^-) \vee (Q^+, Q^-) = (P^+ \vee Q^+,\ P^- \wedge Q^-)$$

- This yields a compositional truthmaker semantics: the semantic value of a sentence $A$ is a (bi)proposition. The value of complex sentences is obtained by these operations.

# States, possibility, and worlds*

- We distinguish a set $S^\Diamond \subseteq S$ of *possible* (consistent) states, assumed **downward closed**:

$$s \in S^\Diamond \mathbin{\&} t \leq s \;\Rightarrow\; t \in S^\Diamond$$

- **Compatibility**: two states $s, t$ are (jointly) *compatible* iff they can co-exist in some possible state

$$s \text{ and } t \text{ compatible} \iff \exists u \in S^\Diamond\, (s \leq u \mathbin{\&} t \leq u)$$

Thus: $s$ and $t$ can both obtain together without inconsistency.

- A *world* is a **maximal possible state**: a state $w \in S^\Diamond$ such that any possible state compatible with $w$ is already part of $w$:

$$w \text{ is a world} \iff w \in S^\Diamond \text{ and } \forall s \in S^\Diamond (s \text{ compatible with } w \Rightarrow s \leq w)$$

# Bilateral propositions and truth / falsity at a world*

Let $w$ be a world (maximal possible state).

**Truth at** $w$:

$$P \text{ is true at } w \iff \exists s \in P^+ \text{ with } s \leq w$$

**Falsity at** $w$:

$$P \text{ is false at } w \iff \exists s \in P^- \text{ with } s \leq w$$

# Exclusivity, exhaustivity, and bivalence*

- **Exclusivity**: no verifier of $P$ is compatible with any falsifier of $P$.

  $$\forall s \in P^+ \, \forall t \in P^- \, \neg(s \text{ compatible with } t)$$

  So at no world $w$ is $P$ both true and false (no gluts).

- **Exhaustivity**: every possible state is compatible with some verifier or some falsifier of $P$.

  $$\forall s \in S^\Diamond \, \exists t \in P^+ \cup P^- \, (s \text{ compatible with } t)$$

  So at every world $w$, $P$ is at least true or false (no gaps).

- Hence, if $P$ satisfies both conditions, then for every world $w$ exactly one of $P$ *is true at* $w$ and $P$ *is false at* $w$ holds: $P$ is bivalent at worlds.

# Extremal propositions*

From the extremal states $0$ and $1$ we get four **extremal unilateral propositions**:

$$T_0 = \{0\} \quad F_0 = \emptyset \quad T_1 = S \quad F_1 = \{1\}$$

- $T_0$: trivially true, *nothing* beyond $0$ is required.
- $T_1$: trivial truth verified by *any* state.
- $F_0$: trivial falsehood, no state can verify it.
- $F_1$: trivial falsehood verified only by the “impossible” full state $1$.

# Two kinds of consequence: entailment and containment*

For verifiable *unilateral* propositions $P, Q \subseteq S$ Fine defines two distinct consequence relations:

- $P$ **entails** $Q$ (notation: $P \leq_d Q$) iff

  $$P \subseteq Q$$

  Every verifier of $P$ is also a verifier of $Q$.
  Disjunction $P \vee Q$ is the *least upper bound* of $P$ and $Q$ w.r.t. $\leq_d$: it is the strongest proposition entailed by both.

- $Q$ **contains** $P$ (notation: $P \leq_c Q$) iff:

  $$\begin{array}{ll} \text{(i)} & \forall q \in Q\ \exists p \in P\ (p \leq q) \\ \text{(ii)} & \forall p \in P\ \exists q \in Q\ (p \leq q) \end{array}$$

  Conjunction $P \wedge Q$ is the *greatest lower bound* of $P$ and $Q$ w.r.t. $\leq_c$ (in a regular domain).

# Consequences for bilateral propositions*

For bilateral $P = (P^+, P^-)$ and $Q = (Q^+, Q^-)$, we define two consequence relations:

- [Containment] $P$ is **contained in** $Q$ iff

$$P^+ \leq_c Q^+ \quad \text{and} \quad P^- \leq_d Q^-$$

- [Entailment] $P$ **entails** $Q$ iff

$$P^+ \leq_d Q^+ \quad \text{and} \quad P^- \leq_c Q^-$$

These two consequence relations interact nicely with negation:

$$P \leq_d Q \Rightarrow \neg Q \leq_c \neg P$$
$$P \leq_c Q \Rightarrow \neg Q \leq_d \neg P$$

# Fine: three notions of verification*

Let $\mathbf{A}$ be the proposition corresponding to $A$. Depending on which closure conditions we impose, $\mathbf{A}$ may be any of

$$|A|, \quad |A|^*, \quad |A|_*, \quad |A|^{**}$$

(no closure; fusion-closed; convex; regular = fusion-closed + convex).

Relative to a model $\mathcal{M}$, Fine distinguishes three verification relations on states $s$:

| | |
|---|---|
| Exact: | $s \models_{\mathsf{ex}} A \iff s \in \mathbf{A}$ |
| Inexact: | $s \models_{\mathsf{in}} A \iff \exists s' \leq s\ (s' \models_{\mathsf{ex}} A)$ |
| Loose (classical): | $s \models A \iff \forall u (u\ comp\ s \Rightarrow \exists t \in \mathbf{A}\ (u\ comp\ t))$ |

where $u\ comp\ s$ means: $u$ is *compatible* with $s$ in the underlying modal state space.

# Fine: notions of consequence*

From these verification relations we obtain corresponding consequence relations (between formulas $A$ and $C$):

| | |
|---|---|
| Exact | $A \models_{\mathsf{e}} C \iff \forall s(s \models_{\mathsf{ex}} A \Rightarrow s \models_{\mathsf{ex}} C)$ |
| Inexact | $A \models_{\mathsf{i}} C \iff \forall s(s \models_{\mathsf{in}} A \Rightarrow s \models_{\mathsf{in}} C)$ |
| Analytic (containment) | $A > C \iff [A] \geq_c [C]$ |
| Loose / classical | $A \models_{\mathsf{l}} C \iff \forall s(s \models A \Rightarrow s \models C)$ |

Each induces a different logic:

- $\models_{\mathsf{l}}$: classical consequence.
- $>$: Angell's analytic entailment.
- $\models_{\mathsf{i}}$: first-degree entailment.
- $\models_{\mathsf{e}}$: exact truthmaker logic (Correia, Fine & Jago).

# Outline

# Exact truthmaking

- On the *necessitation* picture, any state that guarantees $A$ counts as a truthmaker for $A$.
- Often this is too coarse: we want to exclude states with *irrelevant* parts.
- **Exact truthmaker**: a state $s$ which
  - guarantees that $A$ is true, and
  - is wholly relevant to $A$'s truth: no proper part of $s$ is "irrelevant" w.r.t. $A$.
- If $s$ is an exact truthmaker for $A$, then every part of $s$ either belongs to a truthmaker for $A$ or is needed to fuse such parts together.
- Fine & Jago (2019) study the logic of *exact truthmaker semantics*.

# Frames and exact truthmaking models

- A **frame** is a structure

$$\langle S, \leq \rangle$$

  where:

  - $S$ is a non-empty set of *states*
  - $\leq$ is a partial order (parthood / inclusion)
  - every pair $s, t \in S$ has a least upper bound (fusion) $s \sqcup t$

- An **exact truthmaking model** is

$$\mathcal{M} = \langle S, \leq, I \rangle$$

  where $I = (I^+, I^-)$ is a pair of functions $S \times P \to \{0, 1\}$ satisfying $\sqcup$-closure:

  If $I^+(s, p) = 1$ and $I^+(s', p) = 1$, then $I^+(s \sqcup s', p) = 1$

  If $I^-(s, p) = 1$ and $I^-(s', p) = 1$, then $I^-(s \sqcup s', p) = 1$

# Exact truthmaking and falsemaking: clauses

Fix a model $\mathcal{M} = \langle S, \leq, I \rangle$ We define, by recursion on $\varphi$, two relations $\models^+$ and $\models^-$ for positive and negative exact truthmaking:

$s \models^+ p$ iff $I^+(s, p) = 1$
$s \models^- p$ iff $I^-(s, p) = 1$

$s \models^+ \neg\varphi$ iff $s \models^- \varphi$
$s \models^- \neg\varphi$ iff $s \models^+ \varphi$

$s \models^+ \varphi \wedge \psi$ iff there are $s', s'' \in S$ with $s' \sqcup s'' = s$ and $s' \models^+ \varphi$ and $s'' \models^+ \psi$
$s \models^- \varphi \wedge \psi$ iff $s \models^- \varphi$ or $s \models^- \psi$ or there are $s', s'' \in S$ with $s' \sqcup s'' = s$ and $s' \models^- \varphi$ and $s'' \models^- \psi$

$s \models^+ \varphi \vee \psi$ iff $s \models^+ \varphi$ or $s \models^+ \psi$ or there are $s', s'' \in S$ with $s' \sqcup s'' = s$ and $s' \models^+ \varphi$ and $s'' \models^+ \psi$
$s \models^- \varphi \vee \psi$ iff there are $s', s'' \in S$ with $s' \sqcup s'' = s$ and $s' \models^- \varphi$ and $s'' \models^- \psi$

# Logical Consequence[4]

$\Gamma \models_{\mathsf{TM}} \varphi$ iff all models $M$ and states $s$, if $s \models^+ \psi$ for all $\psi \in \Gamma$, then $s \models^+ \varphi$.

[4] We discuss here the *distributive* notion of logical consequence. Fine & Jago (2019) also consider a collective one, which takes the conjunction of all the premises.

# Exactness and the failure of ∧-elimination

- Exactness is built into the clauses for $\models^+$: from

$$s \models^+ p$$

  we do *not* infer that $s \sqcup t \models^+ p$ for arbitrary $t$. Adding irrelevant parts does not preserve exact truthmaking.

- Example: $\mathcal{M} = \langle S, \leq, I \rangle$:
  - States $S = \{s_1, s_2, s_3\}$ with $s_3 = s_1 \sqcup s_2$.
  - Atomic valuation:

$$I^+(s_1, p) = 1 \qquad I^+(s_2, q) = 1 \qquad I^+(s_3, p) = 0$$

- By the conjunction clause for $\models^+$:

$$s_3 \models^+ (p \wedge q)$$

- But $s_3 \not\models^+ p$
- Hence we **lose** ∧-elimination for exact entailment:

$$p \wedge q \;\not\models_{\text{TM}}\; p$$

# Exact disjunction: “exclusive” vs “inclusive”

For conjunction, exact truthmaking works in a purely mereological way:

$$s \models^{+} p \wedge q \;\text{ iff }\; \exists t, u(s = t \sqcup u,\; t \models^{+} p,\; u \models^{+} q)$$

For disjunction there are two natural options.

- **Exclusive disjunction:**

  $$s \models^{+}_{\text{exc}} p \vee q \;\text{ iff }\; s \models^{+} p \text{ or } s \models^{+} q$$

  Only states that already make one disjunct true themselves count as exact truthmakers for $p \vee q$.

  $p \wedge q \not\models_{\text{TM}} p \vee q$

- **Inclusive disjunction:**

  $$s \models^{+}_{\text{inc}} p \vee q \;\text{ iff}$$
  $$s \models^{+} p \text{ or } s \models^{+} q$$
  $$\text{or } \exists t, u(s = t \sqcup u,\; t \models^{+} p,\; u \models^{+} q)$$

  $p \wedge q \models_{\text{TM}} p \vee q$

# In-class examples

- Examples and models on blackboard
- $(p \vee q) \wedge (p \vee r) \not\models_{\mathsf{TM}} p \vee (q \wedge r)$
- $(p \wedge q) \vee p \not\models_{\mathsf{TM}} p$

The latter also shows that an exact verifier does not have to be minimal:

$s$ minimally verifies $\varphi$ if $s \models^+ \varphi$ and for any $s' \leq s$ s.t. $s' \models^+ \varphi$, $s' = s$.

# From exact truthmaking to van Fraassen's primary bases

For each formula $A$ define its exact truthmaking and falsemaking sets:

$$|A|^+ := \{s \in S : s \models^+ A\} \qquad |A|^- := \{s \in S : s \models^- A\}$$

Using the clauses and fusion-closure of $I^\pm$, one shows by induction on $A$ that $|A|^+$ and $|A|^-$ are closed under finite fusions $s \sqcup t$.

We take the semantic clauses in their **exclusive** version, and we now recover van Fraassen's primary $T$- and $F$-bases from $|A|^+$ and $|A|^-$.

**Fact-like assumption:** For each propositional atom $p$ there are distinguished *atomic* states $e_p, \overline{e_p} \in S$ such that $|p|^+ = \{e_p\}$ and $|p|^- = \{\overline{e_p}\}$

# From exact truthmaking to van Fraassen's primary bases

For every formula $A$ set

$$T(A) := |A|^+ \qquad F(A) := |A|^-$$

By induction on $A$, the $T(A), F(A)$ just defined satisfy van Fraassen's recursive primary bases:

$$
\begin{aligned}
T(p) &= \{e_p\} & F(p) &= \{\overline{e_p}\} \\
T(\neg A) &= F(A) & F(\neg A) &= T(A) \\
T(A \wedge B) &= \{s \sqcup t : s \in T(A),\ t \in T(B)\} & F(A \wedge B) &= F(A) \cup F(B) \\
T(A \vee B) &= T(A) \cup T(B) & F(A \vee B) &= \{s \sqcup t : s \in F(A),\ t \in F(B)\}
\end{aligned}
$$

van Fraassen's (inexact) propositions are the closures $\mathsf{T}^*(A) = \mathrm{cl}(T(A))$, $\mathsf{F}^*(A) = \mathrm{cl}(F(A))$ as before.

# Metalogical features

- Although the exact truthmaker consequence relation is unusual, it is very well behaved:
  - **Compact**: if $\Gamma \models_{\mathsf{TM}} \varphi$, then some finite $\Delta \subseteq \Gamma$ already satisfies $\Delta \models_{\mathsf{TM}} \varphi$.
  - **Decidable**: there is an effective procedure to determine, for finite $\Gamma$, whether $\Gamma \models_{\mathsf{TM}} \varphi$.
- Fine & Jago (2019) also provide a sequent calculus for $\models_{\mathsf{TM}}$, which is sound, complete, and admits cut-elimination.

# Application: Imperatives (Fine 2017)*

- There is an intuitive notion of *imperative consequence*:
  - From *Turn on the light and shut the door* one can infer *Turn on the light*.
- Naive idea: reduce imperative consequence to indicative consequence:
  - Let $X$ correspond to an indicative $A$, and $Y$ to $B$.
  - Say: $Y$ follows from $X$ iff $B$ follows (classically) from $A$.
- But this gives Ross's paradox:
  - Indicative: from *You turn on the light* we can infer *You turn on the light or burn the building down*.
  - Imperative: from *Turn on the light* we *cannot* infer *Turn on the light or burn the building down*.
- Truthmaker semantics suggests a different route: give a *direct* semantics for imperatives in terms of actions, parallel to the semantics for indicatives in terms of states.

# Imperatives: exact compliance*

- Extend the state space so that **actions** are a special kind of state.
- For an imperative sentence $X$:
  - An action $\alpha$ is in *exact compliance* with $X$ (intuitively: exactly doing what $X$ requires, no more and no less).
  - An action $\alpha$ is in *exact contravention* of $X$ (intuitively: exactly violating $X$).
- Analogy with indicatives:
  - Indicative: $s$ exactly verifies *You shut the door* iff $s$ is *just* your shutting the door.
  - Imperative: $\alpha$ is in exact compliance with *Shut the door* iff $\alpha$ is *just* your shutting the door (shutting the door *and* turning on the light is not exact compliance).

# Imperative content*

- For a conjunctive imperative $X \wedge Y$:

  $$\alpha \text{ is in exact compliance with } X \wedge Y$$

  iff there are actions $\beta, \gamma$ such that:

  1. $\alpha = \beta \sqcup \gamma$
  2. $\beta$ exactly complies with $X$
  3. $\gamma$ exactly complies with $Y$

- **Content of an imperative** $X$:

  $$|X|^{\mathsf{imp}} = \{\alpha : \alpha \text{ is in exact compliance with } X\}$$

# Imperative consequence as necessary means*

- Given contents $|X|^{\mathsf{imp}}$ and $|Y|^{\mathsf{imp}}$, Fine defines **imperative consequence** by a *containment* (or necessary-means) condition.
- Obeying $X$ always involves, and can be decomposed into, obeying $Y$ plus (possibly) more.
- Formally, $Y$ follows from $X$ iff the content of $Y$ is a *conjunctive part* of the content of $X$:
  - (i) For every $\alpha \in |X|^{\mathsf{imp}}$, there is some $\beta \in |Y|^{\mathsf{imp}}$ with $\beta \leq \alpha$.
  - (ii) For every $\beta \in |Y|^{\mathsf{imp}}$, there is some $\alpha \in |X|^{\mathsf{imp}}$ with $\beta \leq \alpha$.
- Then $Y$ is a *necessary means* to $X$.
- This dissolves Ross's paradox:
  - The content of *Turn on the light or burn the building down* is not a conjunctive part of the content of *Turn on the light*. So the imperative inference fails.
  - But the content of *Turn on the light is* a conjunctive part of the content of *Turn on the light and shut the door*. So that inference is validated.